

On electromagnetic energy in Bardeen and ABG spacetimes

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Abstract

We demonstrate that the total energy of electromagnetic field in the Bardeen and Ayón-Beato-García singularity-free models is equal to the mass parameter M , being therefore independent of the charge parameter Q . Our result is fully congruent with the original idea of Born and Infeld to use nonlinear electrodynamics for proving the electromagnetic nature of mass.

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I. INTRODUCTION

The construction of regular black hole models was pioneered by Bardeen [1] who ingeniously modified the well-known Reissner-Norström metric [2, 3] for spherically symmetric mass and charge to remove the singularity at $r = 0$. Though the global mathematical properties of Bardeen's spacetime are well understood (see, e.g., [4]), the corresponding electromagnetic source for this spacetime was unknown for many years, as the Bardeen model did not originally arise as a solution to some field equations. The first exact regular black hole solutions were constructed, within the framework of Einstein's gravity coupled to nonlinear electrodynamics, by Ayón-Beato and García [5, 6] who also later reinterpreted Bardeen's model as an exact solution for a nonlinear magnetic monopole [7]. Despite a considerable attention these solutions have received in recent years, it seems that the main physical question concerning the Bardeen and ABG spacetimes – How can an arbitrarily small charge remove the physical Schwarzschild singularity of a collapsed star with enormous mass? – still has not been clarified so far. Being strongly convinced that the answer to this question must be closely related to the issue of electromagnetic energy associated with the above spacetimes, in the present letter we will calculate the total electromagnetic energy in the Bardeen and ABG models to reveal that for all these models it has the same value that does not actually depend on the charge parameter Q .

II. THE TOTAL ENERGY OF ELECTRIC FIELD IN ABG SOLUTIONS

We start our consideration with the first ABG solution [5] defined by the metric

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2, \quad f = 1 - \frac{2Mr^2}{(r^2 + Q^2)^{3/2}} + \frac{Q^2 r^2}{(r^2 + Q^2)^2}, \quad (1)$$

and the associated electric field

$$E = Qr^4 \left(\frac{r^2 - 5Q^2}{(r^2 + Q^2)^4} + \frac{15}{2} \frac{M}{(r^2 + Q^2)^{7/2}} \right), \quad (2)$$

the parameters M and Q standing for the mass and electric charge of the source, respectively.

For a static observer, $u^\alpha = (-g_{tt})^{-1/2} \xi^\alpha$, ξ^α being the timelike Killing vector, the energy density of the electromagnetic field is defined as

$$T_{\alpha\beta} u^\alpha u^\beta = (-g_{tt})^{-1} T_{tt} = -T_t^t, \quad (3)$$

where $T_{\alpha\beta}$ is the electromagnetic energy-momentum tensor, so that the quantity $-T_t^t$ can be calculated either via the construction of the tensor $T_{\alpha\beta}$ from the corresponding tensor $F_{\alpha\beta}$ of nonlinear electrodynamics or, more directly, from the Einstein equations,

$$T_{\alpha\beta} = \frac{1}{8\pi} G_{\alpha\beta}. \quad (4)$$

The total electric or magnetic energy, which is of interest to us in this letter, then will be equal to the integral over the surface $t = \text{const}$,

$$\mathcal{E}_{e/m} = \int_{R^3} (-T_t^t) \sqrt{-g} dr d\vartheta d\varphi, \quad (5)$$

where $\sqrt{-g} = r^2 \sin \vartheta$ for all the spherically symmetric spacetimes to be considered.

Let us first obtain the electric energy density for the ABG solution (1) straightforwardly from (4). Then we get[12]

$$-T_t^t = -\frac{1}{8\pi} G_{tt} g^{tt} = \frac{Q^2(r^2 - 3Q^2 + 6M\sqrt{r^2 + Q^2})}{8\pi(r^2 + Q^2)^3}, \quad (6)$$

and it can be shown that the density is a positive definite function if $2M > |Q|$.

While evaluating the total electric energy of the ABG solution (1) by means of formula (5), we find it instructive to carry out the integration over r on the interval $[0, r]$, thus getting $\mathcal{E}_e(r)$, and then tend r to infinity. Therefore, taking into account (6), we obtain

$$\begin{aligned} \mathcal{E}_e(r) &= \int_0^r \int_0^\pi \int_0^{2\pi} (-T_t^t) r^2 \sin \vartheta dr d\vartheta d\varphi \\ &= -\frac{Q^2 r^3}{2(r^2 + Q^2)^2} + \frac{Mr^3}{(r^2 + Q^2)^{3/2}}, \end{aligned} \quad (7)$$

whence it is fairly well clear how in the limit $r \rightarrow \infty$ vanishes the first term on the right-hand side of (7), with Q as a factor, while the second term leads to

$$\mathcal{E}_e(\infty) = M. \quad (8)$$

Of course, one would come to the same result for \mathcal{E}_e if one calculates the component T_t^t not by means of the Einstein tensor (4) but directly from the energy-momentum tensor of electric field defined in [5] as

$$4\pi T_\beta^\alpha = \mathcal{H}_P P_{\beta\mu} P^{\alpha\mu} - \delta_\beta^\alpha (2P\mathcal{H}_P - \mathcal{H}). \quad (9)$$

Indeed, taking into account that for the ABG solution (1)

$$\begin{aligned} P_{\alpha\beta} &= 2\delta_{[\alpha}^t\delta_{\beta]}^r\frac{Q}{r^2}, \quad P^{\alpha\beta} = -2\delta_t^{[\alpha}\delta_r^{\beta]}\frac{Q}{r^2}, \quad P = -\frac{Q^2}{2r^4}, \\ \mathcal{H}_P &= \frac{r^6(2r^2 - 10Q^2 + 15M\sqrt{r^2 + Q^2})}{2(r^2 + Q^2)^4}, \\ \mathcal{H} &= -\frac{Q^2(r^2 - 3Q^2 + 6M\sqrt{r^2 + Q^2})}{2(r^2 + Q^2)^3}, \end{aligned} \quad (10)$$

it is easy to check that (9) and (10) yield the same expression for the energy density as in (6), and consequently the same value of the total electric energy (8).

To be sure that the parameter M in (8) is the ADM mass [9] of the ABG solution, let us consider the Komar [10] mass function $M_K(r)$ defined by the following integral of the 2-form $\omega = -\frac{1}{2}\eta_{\alpha\beta\nu\mu}\nabla^\nu\xi^\mu dx^\alpha \wedge dx^\beta$:

$$M_K(r) = \frac{1}{4\pi} \int_{S_r} \omega, \quad (11)$$

which represents the “mass” inside a sphere of radius r , so that the ADM mass will correspond to $M_K(\infty)$. In the case of the metric (1), ω takes the form

$$\omega = \frac{1}{2}\omega_{\vartheta\varphi}d\vartheta \wedge d\varphi, \quad (12)$$

with

$$\omega_{\vartheta\varphi} = -\frac{2r^3[Q^2(r^2 - Q^2) - M(r^2 - 2Q^2)\sqrt{r^2 + Q^2}]\sin\vartheta}{(r^2 + Q^2)^3}, \quad (13)$$

and thus we have

$$\begin{aligned} M_K(r) &= \frac{1}{8\pi} \int_0^\pi \int_0^{2\pi} \omega_{\vartheta\varphi} d\vartheta d\varphi = \frac{1}{4} \int_0^\pi \omega_{\vartheta\varphi} d\vartheta \\ &= -\frac{r^3[Q^2(r^2 - Q^2) - M(r^2 - 2Q^2)\sqrt{r^2 + Q^2}]}{(r^2 + Q^2)^3}, \end{aligned} \quad (14)$$

whence, in the limit $r \rightarrow \infty$, we finally arrive at

$$M_K(\infty) = M. \quad (15)$$

Therefore, the total electric energy of the solution (1) is equal to the ADM mass M independently of the value of the charge parameter Q . Though this result may look surprising at first glance, it nevertheless is quite logic as it leaves no doubt that the electric energy in the metric (1) is comparable with the ADM mass and hence seems to be able to regularize the Schwarzschild singular spacetime in principle. At the same time, it is also clear that

the ABG solution (1) can hardly describe the field of a point charge, but rather of some distribution of positive and negative charges for which the particular value of Q , playing in such a case the role of a net charge, does not really matter. In Fig. 1 we have plotted the functions $\mathcal{E}_e(x)/M$ and $M_K(x)/M$ of this solution versus the dimensionless variable $x = r/|Q|$. Note also that the total electric energy corresponding to the “massless” ($M = 0$) subfamily of the metric (1) is zero for any Q , which is an indication that this one-parameter spacetime must have regions of positive and negative energy.

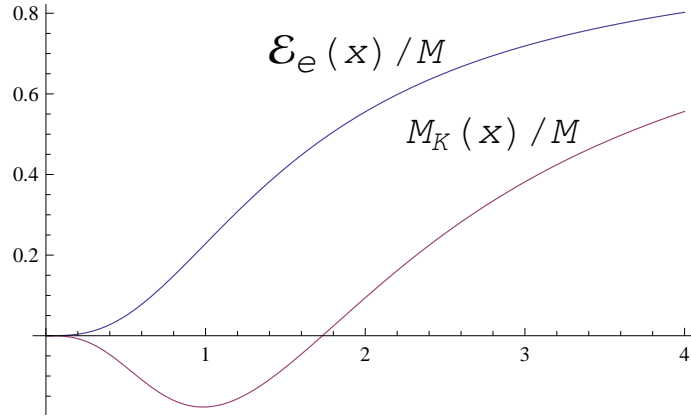


FIG. 1: Behavior of the functions $\mathcal{E}_e(x)/M$ and $M_K(x)/M$, with $x = r/|Q|$ and $|Q|/M = 1$, in the case of the first ABG solution.

A. The second ABG solution

It turns out that the above said about the energy of the ABG metric (1) is fully applicable to another ABG spacetime described by the metric [6]

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2, \quad f = 1 - \frac{2M}{r} \left(1 - \tanh \frac{Q^2}{2Mr} \right), \quad (16)$$

and the electric potential

$$E = \frac{Q}{4Mr^3} \left(1 - \tanh^2 \frac{Q^2}{2Mr} \right) \left(4Mr - Q^2 \tanh \frac{Q^2}{2Mr} \right). \quad (17)$$

Indeed, like in the previous case, the density of electric field can be evaluated through the Einstein tensor, yielding

$$-T_t^t = \frac{Q^2}{8\pi r^4} \operatorname{sech}^2 \frac{Q^2}{2Mr}, \quad (18)$$

which is a positive definite function for any nonzero values of M and Q . Then the electric energy contained inside a sphere of radius r is given by the expression

$$\mathcal{E}_e(r) = M - M \tanh \frac{Q^2}{2Mr}, \quad (19)$$

and, for large r , it behaves as $M - \frac{Q^2}{2r} + O\left(\frac{1}{r^2}\right)$, so that for the total energy of electric field $\mathcal{E}_e(\infty)$ we again obtain, after taking the limit $r \rightarrow \infty$ in (19), the value M .

The Komar mass function $M_K(r)$ of the second ABG solution is determined by the formulas (11) and (12) with

$$\omega_{\vartheta\varphi} = \sin \vartheta \left[2M \left(1 - \tanh \frac{Q^2}{2Mr} \right) - \frac{Q^2}{r} \operatorname{sech}^2 \frac{Q^2}{2Mr} \right], \quad (20)$$

and therefore, taking into account (14), we get

$$M_K(r) = M - \frac{Q^2}{r \left(1 + \cosh \frac{Q^2}{Mr} \right)} - M \tanh \frac{Q^2}{2Mr}. \quad (21)$$

Then it follows from (21) that the ADM mass of this solution is $M_K(\infty) = M$, and one can also verify that $M_K(r)$ vanishes at $r = 0$. The characteristic behavior of the functions $\mathcal{E}_e(r)$ and $M_K(r)$ in the vicinity of $r = 0$ is shown in Fig. 2, where we introduced the dimensionless variable $x = Mr/Q^2$ and divided those functions by M for obtaining generic plots not depending on concrete values of M and Q . There, one can observe the presence of the region with negative values of the Komar function.

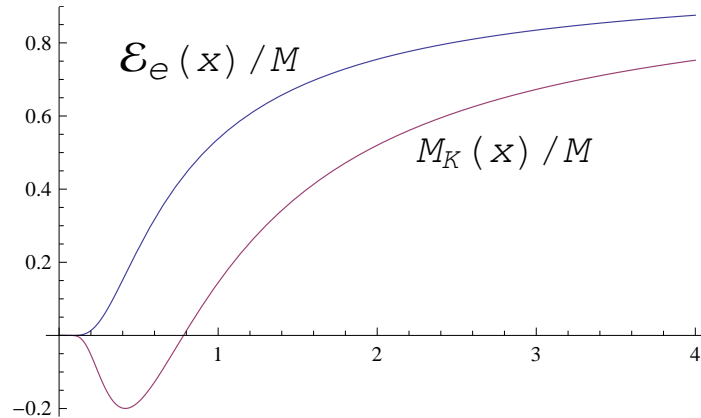


FIG. 2: Behavior of the functions $\mathcal{E}_e(x)/M$ and $M_K(x)/M$, with $x = Mr/Q^2$, in the case of the second ABG solution for arbitrary nonzero M and Q .

III. THE TOTAL ENERGY OF MAGNETIC FIELD IN BARDEEN SPACETIME

We now turn to analyzing the electromagnetic energy issue in Bardeen's spacetime given by the metric [1]

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2, \quad f = 1 - \frac{2Mr^2}{(r^2 + Q^2)^{3/2}}, \quad (22)$$

in which the parameter Q was originally interpreted as describing the electric charge, but later reinterpreted by Ayón-Beato and García [7] as representing a nonlinear magnetic monopole with the electromagnetic tensor

$$F_{\alpha\beta} = 2\delta_{[\alpha}^{\vartheta} \delta_{\beta]}^{\varphi} Q \sin \vartheta. \quad (23)$$

Once again choosing the most convenient way of finding the density of electromagnetic field solely through the metric (22), we readily obtain

$$-T_t^t = \frac{3MQ^2}{4\pi(r^2 + Q^2)^{5/2}}, \quad (24)$$

so that the magnetic energy $\mathcal{E}_m(r)$ inside a sphere of radius r will have the form

$$\mathcal{E}_m(r) = \frac{MR^3}{(r^2 + Q^2)^{3/2}}, \quad (25)$$

thus leading in the limit $r \rightarrow \infty$ to the expectable result for the total energy of magnetic field:

$$\mathcal{E}_m(\infty) = M. \quad (26)$$

As for the Komar mass function associated with the Bardeen spacetime, it is obtainable from (11), (12) and (14) taking into account that

$$\omega_{\vartheta\varphi} = \frac{2Mr^3(r^2 - 2Q^2) \sin \vartheta}{(r^2 + Q^2)^{5/2}}, \quad (27)$$

hence yielding

$$M_K(r) = \frac{Mr^3(r^2 - 2Q^2)}{(r^2 + Q^2)^{5/2}}. \quad (28)$$

As a result, the ADM mass $M_K(\infty)$ of this spacetime is equal to M , similar to the two ABG solutions previously considered.

It follows from (28) that the Komar function of Bardeen's model takes negative values on the interval $0 < r < \sqrt{2}|Q|$ (of course, we assume that $M > 0$), and it has one minimum

at $r_m = \sqrt{\frac{2}{3}}|Q|$, so that for $r > r_m$, $M_K(r)$ is an increasing function. Note that although the functions $\mathcal{E}_m(r)$ and $M_K(r)$ in (25) and (28) differ from the respective expressions in the ABG solutions, still their behavior in Bardeen's case depicted in Fig. 3 is very similar to that shown earlier in Figs. 1 and 2.

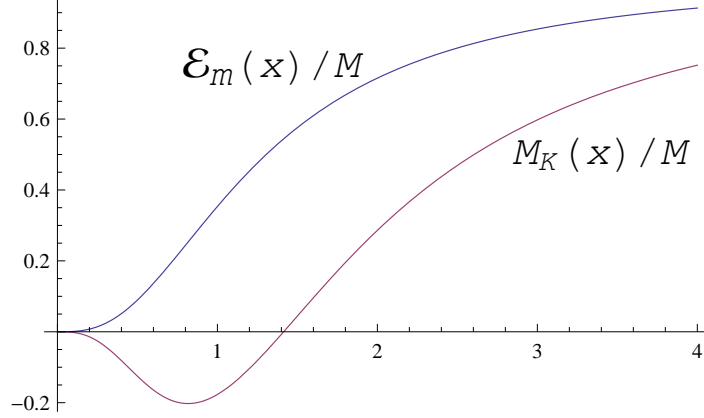


FIG. 3: Behavior of the functions $\mathcal{E}_m(x)/M$ and $M_K(x)/M$, with $x = r/|Q|$, in the case of the Bardeen spacetime for arbitrary nonzero M and Q .

It is worth mentioning that in the case of the Schwarzschild solution the component $\omega_{\vartheta\varphi}$ of the 2-form ω is equal to $2M \sin \vartheta$, being independent of r , and consequently $M_K(r) = M$ for any $r > 0$, which means that the whole mass of the Schwarzschild black hole is contained in the singularity at $r = 0$. In this respect, it appears that the mass in the Bardeen and ABG models is not localized in some restricted region but rather is distributed over the entire space.

Let us also note for completeness that in the Reissner-Nordström solution, which is singular at $r = 0$, the expression for the density of electric field does not involve the mass parameter M , being equal to $Q^2/8\pi r^4$. This implies that the corresponding expression of the electric energy is independent of M too; and although (as is well known) the respective integral over the whole space is divergent, still the integration over r makes sense on the interval $[r, +\infty)$, $r > 0$, giving $Q^2/2r$. The analogous energy of magnetic field in Bardeen's model on the latter interval is equal to $M - Mr^3(r^2 + Q^2)^{-3/2}$, and it vanishes when either of the parameters M or Q is equal to zero.

IV. CONCLUDING REMARKS

It is really surprising that all three different models of non-singular black-hole spacetimes considered in the present paper share the same fundamental characteristic with regard to the issue of the total electromagnetic energy whose value, on the one hand, turns out to be independent of the charge parameter Q and, on the other hand, is equal exactly to the ADM mass M . At the same time, this result strongly suggests that, from the global point of view, the entire “mass” in the Bardeen and ABG models comes from the electromagnetic field and the particular values of Q do not affect it. Indeed, after converting the Schwarzschild singularity (that contained the whole mass) into a regular mass distribution by means of nonlinear electrodynamics, one is obliged to explain the origin of that novel mass distribution through the corresponding energy-momentum tensor. So, when the latter tensor is that of the electromagnetic field only, with no any other sources of gravity, then one inevitably arrives at the conclusion that the mass in such regular spacetimes must have the electromagnetic origin. In this respect, it would be worth recalling the original paper of Born and Infeld [11] in which the modified Maxwell’s equations had been used for deducing the electromagnetic origin of inertia, and we have an impression that in the papers [5–7] this old idea contradicting the modern conception about the nature of mass was just reproduced at a new level.

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